MATH 512 HOMEWORK 2

Due Wednesday, March 6

Problem 1. *Suppose that* κ *is measurable and* U *is a normal measure on* κ*. Show that* $\{\alpha < \kappa \mid \alpha \text{ is an inaccessible cardinal}\} \in U$*. Also show that if* $\{\tau < \kappa \mid 2^\tau \leq \tau^{++}\} \in U$*, then* $2^\kappa \leq \kappa^{++}$*.*

Recall that κ has the tree property if every tree with height κ and levels of size less than κ has an unbounded branch.

Problem 2. *Show that if* κ *is measurable, then it has the tree property. In particular, measurable cardinals are weakly compact.*
Hint: given $j : V \to M$ *and a tree* T *with height* κ *and levels of size less than* κ*, look at a node on the* κ*-th level of* $j(T)$*.*

Problem 3. *Suppose that* κ *is measurable and* U_1, U_2 *are two normal measures on* κ *such that* $U_1 \in Ult(V, U_2)$*. I.e.* $U_1 < U_2$ *in the Mitchell order. Show that* $\{\tau < \kappa \mid \tau \text{ is a measurable cardinal}\}$ *is stationary.*

Recall that κ is strongly compact if for every S, every κ-complete filter on S can be extended to a κ-complete ultrafilter.

Problem 4. *Show that if* κ *is strongly compact, then it is measurable.*

Recall that an algebra $\mathcal{B} \subset \mathcal{P}(\kappa)$ is κ-complete if whenever $\langle A_\alpha \mid \alpha < \tau \rangle$ are sets in $\mathcal{B}$ for some $\tau < \kappa$, then so is $\bigcap_{\alpha<\tau} A_\alpha$.

Problem 5. *Show that the following are equivalent for a cardinal* κ*:*

1. κ *is inaccessible and has the tree property;*
2. κ *is inaccessible and for every* κ*-complete algebra* $\mathcal{B} \subset \mathcal{P}(\kappa)$ *of size* κ*, every* κ*-complete filter on* $\mathcal{B}$ *can be extended to a* κ*-complete ultrafilter.*

Each of the above gives a characterization for weak compactness.

Remark 1. A third characterization of weak compactness is the following: κ is inaccessible and $\mathcal{L}_{\kappa,\omega}$ satisfies the Weak Compactness Theorem.

Here the language $\mathcal{L}_{\kappa,\omega}$ contains κ variables, the usual logical connectives and quantifiers, and infinitary connectives $\bigvee_{\alpha<\tau} \phi_\alpha$, $\bigwedge_{\alpha<\tau} \phi_\alpha$ for any $\tau < \kappa$ (infinite disjunction and conjunction of size less than κ). $\mathcal{L}_{\kappa,\omega}$ satisfies the weak Compactness Theorem if for every set of sentences $\Sigma \subset \mathcal{L}_{\kappa,\omega}$ with $|\Sigma| \leq \kappa$, if every $S \subset \Sigma$ with $|S| < \kappa$ has a model, then Σ has a model.